Spring 2012

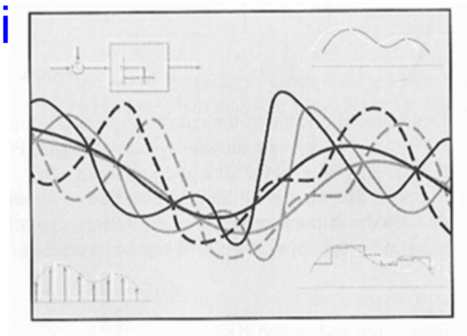
Signals and Systems

Chapter SS-7
Sampling

Shou shui Wei

SDU-BME

Sep08 – Dec08



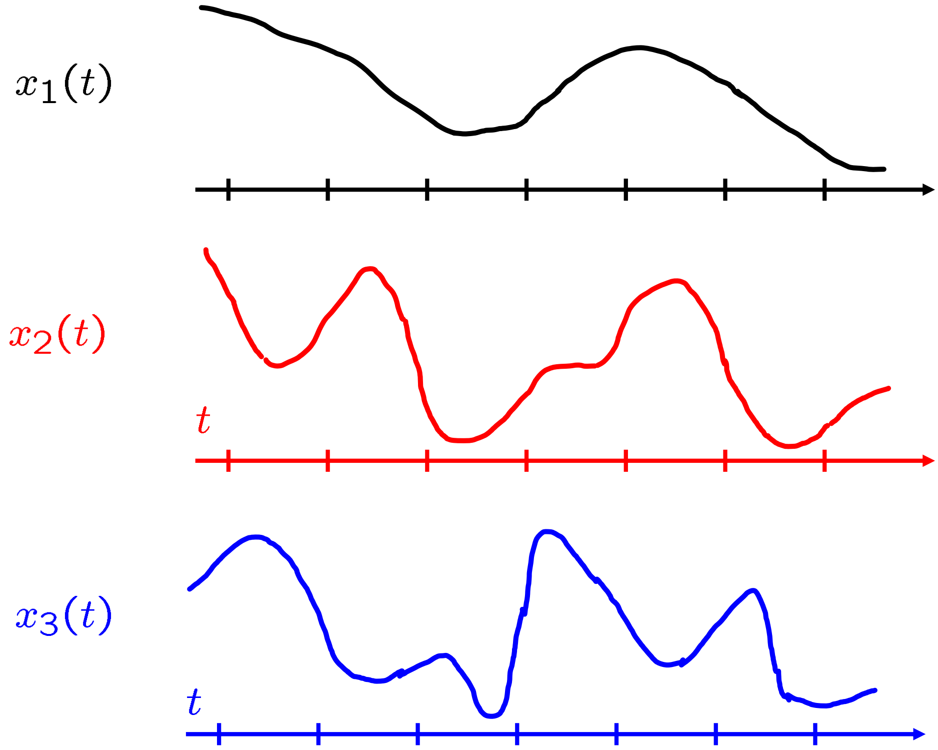
Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

Outline

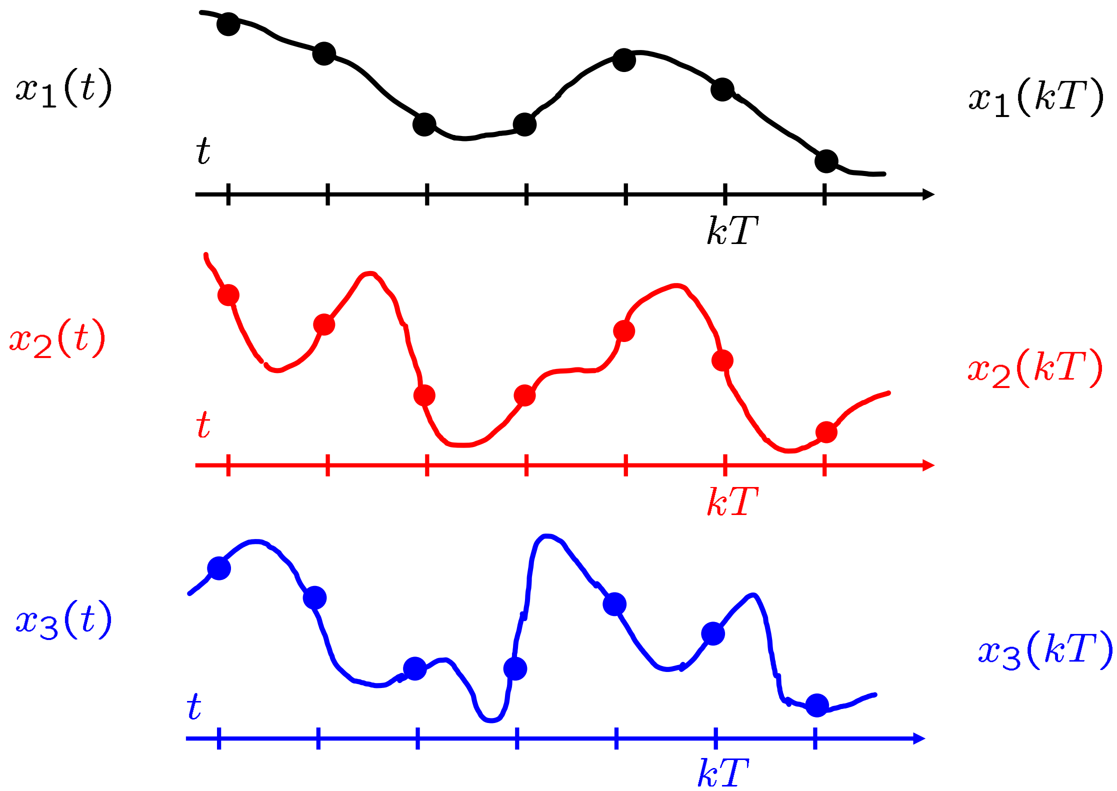
Shou shui Wei©2012

- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

- Representation of CT Signals by its Samples



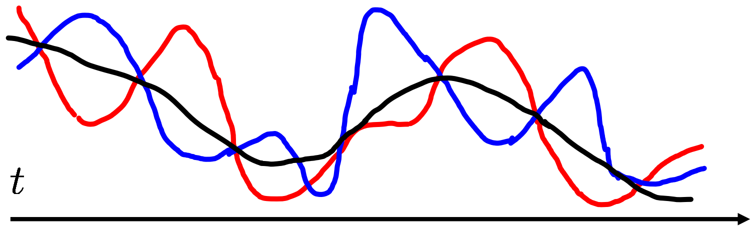
- Representation of CT Signals by its Samples



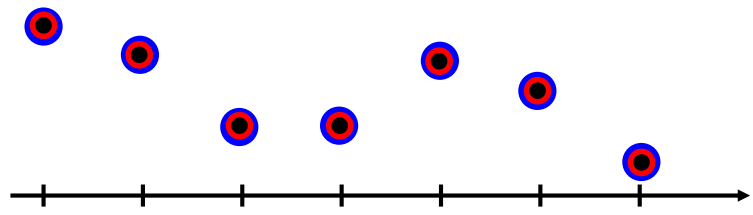
Shou shui Wei©2012

Representation of CT Signals by its Samples

$x_1(t) \neq x_2(t) \neq x_3(t)$



$x_1(kT) = x_2(kT) = x_3(kT)$



Shou shui Wei©2012

Impulse-Train Sampling:

$p(t)$: sampling function

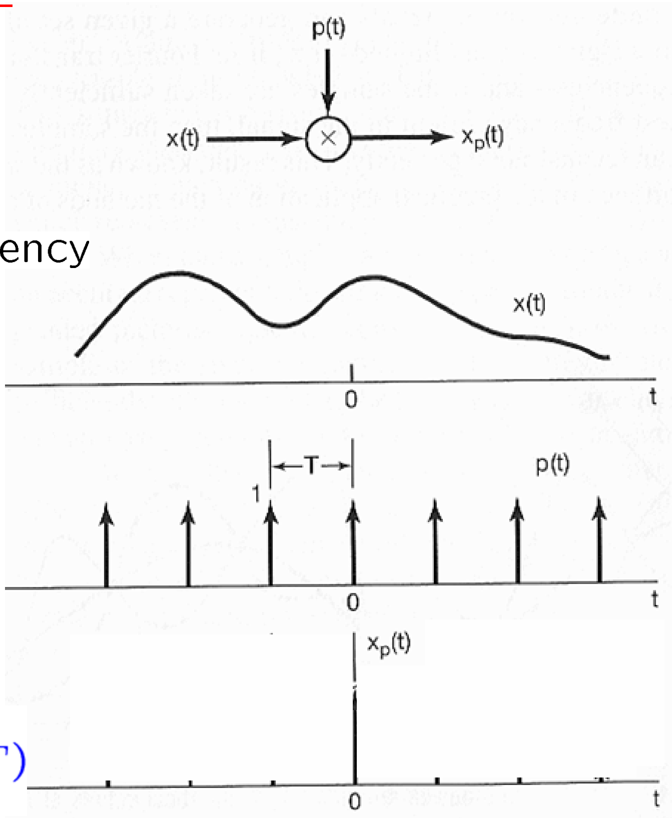
T : sampling period

$w_s = \frac{2\pi}{T}$: sampling frequency

$\Rightarrow x_p(t) = x(t)\, p(t)$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$



- **Impulse-Train Sampling:** From multiplication property,

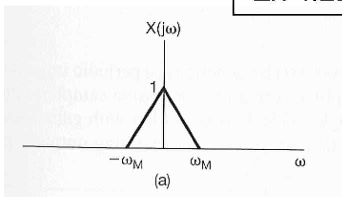
$$x_p(t) = x(t)\, p(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_p(jw) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)P(j(w-\theta))d\theta$$

Eq 4.70, p. 322

$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j(w-kw_s))$$

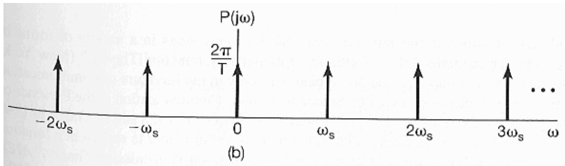
Ex 4.21, p. 323

$$x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(jw)$$



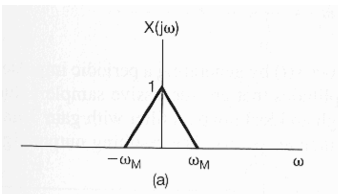
$$p(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad P(jw) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(w-kw_s)$$

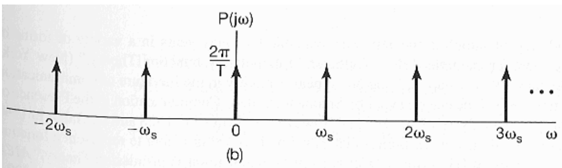
Ex 4.8, pp. 299-300



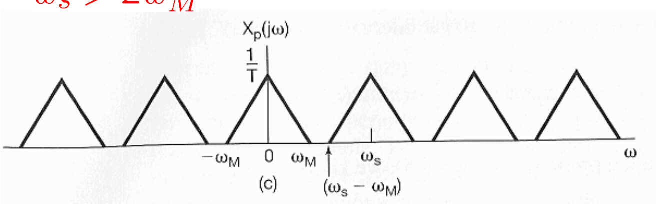
- **Impulse-Train Sampling:**

Ex 4.21, 4.22, pp. 323-4

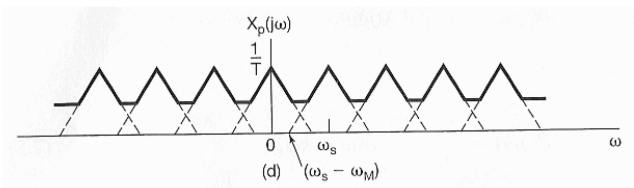




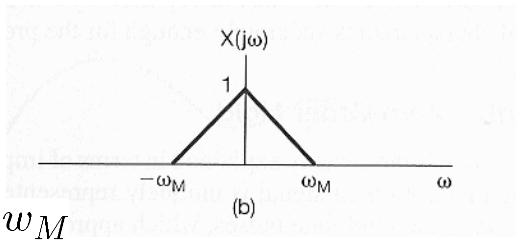
$w_s > 2w_M$



$w_s < 2w_M$



▪ The Sampling Theorem:

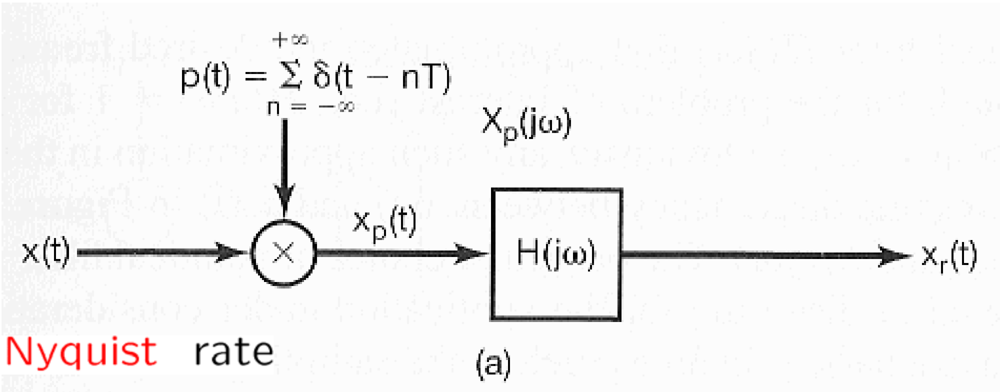


$x(t)$: a band-limited signal

with $X(jw) = 0$ for $|w| > w_M$

if $w_s > 2w_M$ where $w_s = \dfrac{2\pi}{T}$

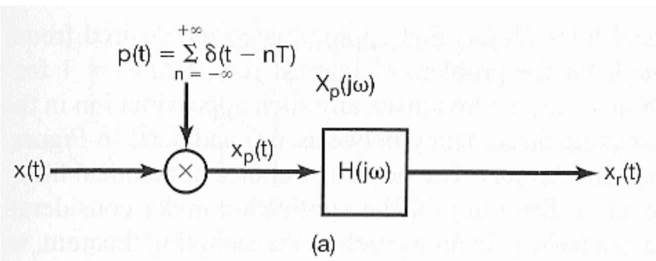
$\Rightarrow$ $x(t)$ is uniquely determined by $x(nT), n = 0, \pm 1, \pm 2, ...,$

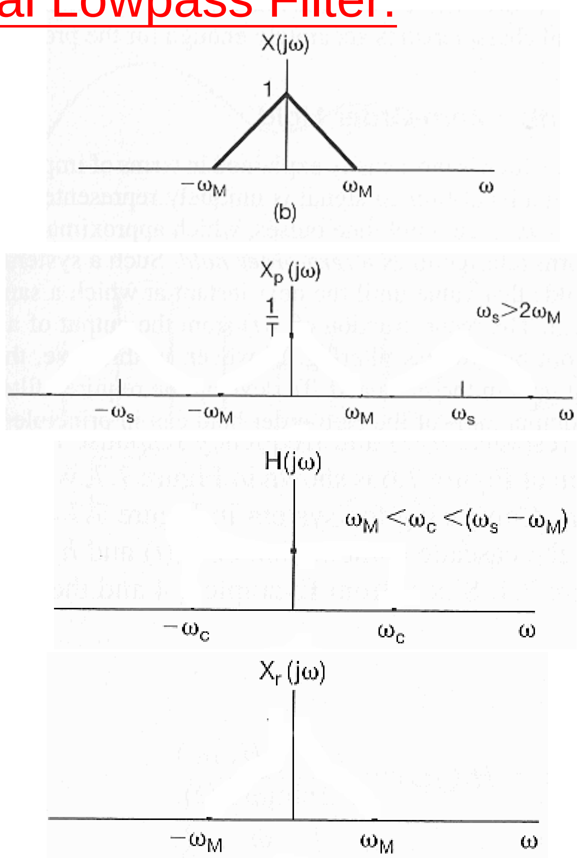


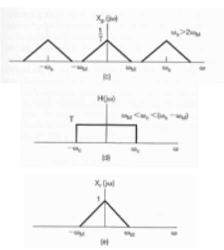
$\Rightarrow$ $2w_M$: Nyquist rate

w_M : Nyquist frequency

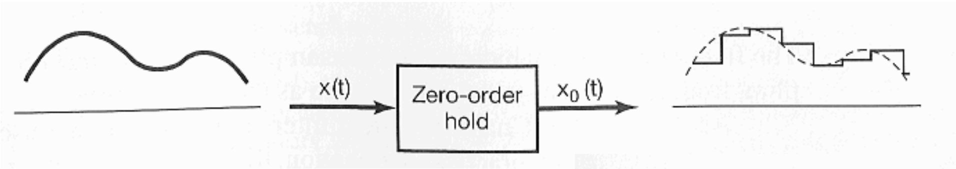
▪ Exact Recovery by an Ideal Lowpass Filter:

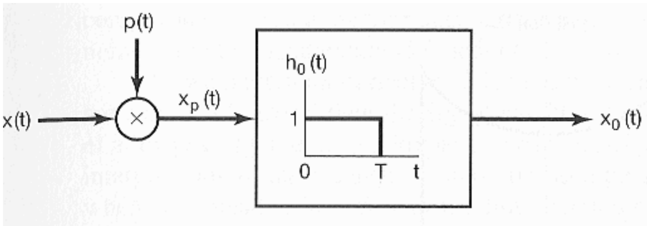


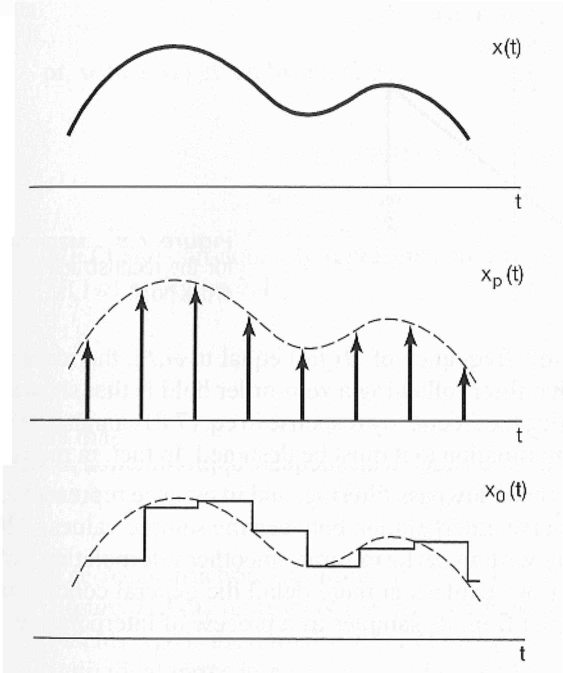




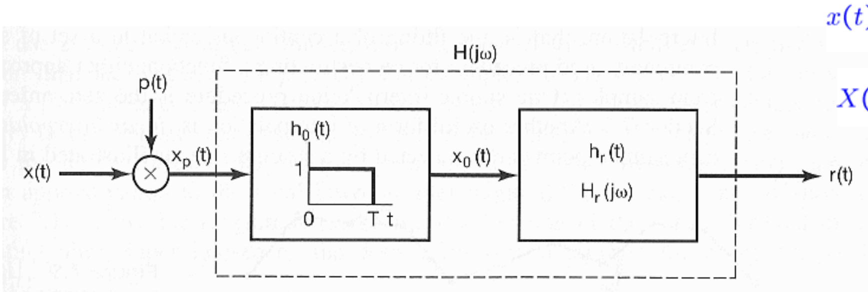
- **Sampling with Zero-Order Hold:**





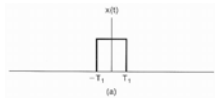


- **Sampling with Zero-Order Hold:**

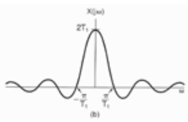


Ex 4.4, p. 293

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



$$X(jw) = 2\frac{\sin(wT_1)}{w}$$



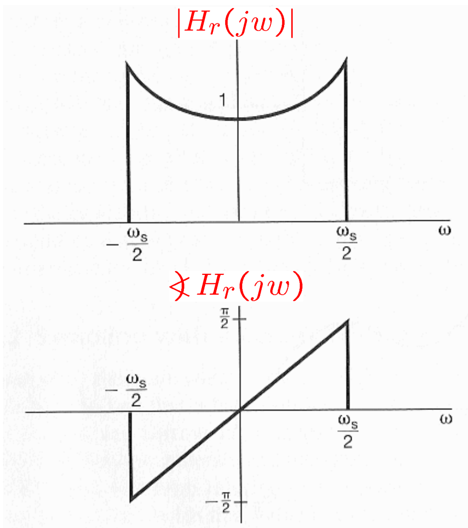
Eq 4.27, p. 301

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}X(jw)$$

$$H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$

$$H(jw) = H_0(jw)H_r(jw)$$

$$\Rightarrow H_r(jw) = \frac{e^{jwT/2}H(jw)}{\frac{2\sin(wT/2)}{w}}$$



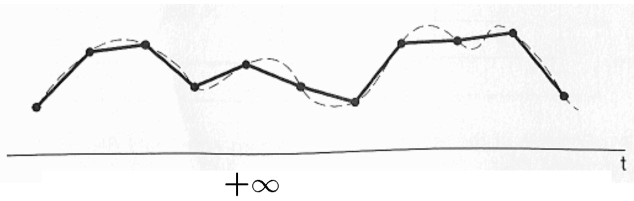
Outline

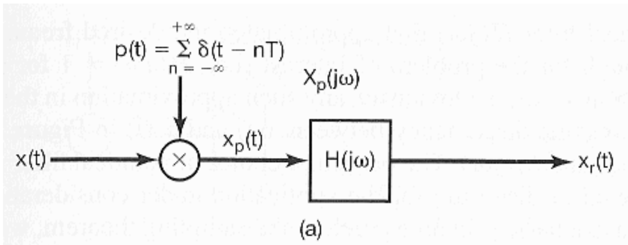
- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

Reconstruction of a Signal from its Samples Using Interpolation

Shou shui Wei©2012

- Exact Interpolation:





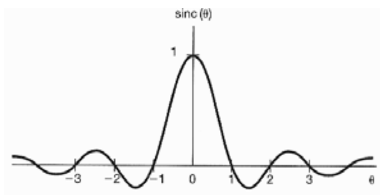
$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

ideal lowpass filter with a magnitude of T

$$x_r(t) = x_p(t) * h(t)$$

$$h(t) = T\,\frac{w_c}{\pi}\,\frac{\sin(w_c t)}{w_c t}$$

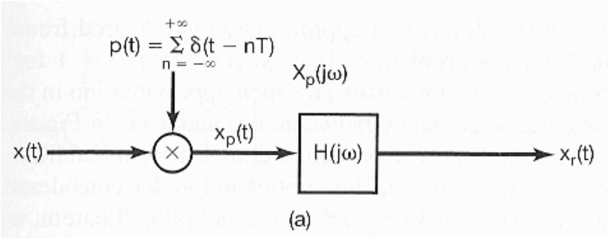
Ex 2.11, p. 110 $\quad x(t - t_0) = x(t) * \delta(t - t_0)$

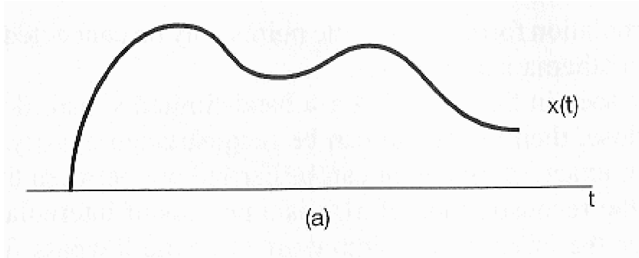


$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)$$

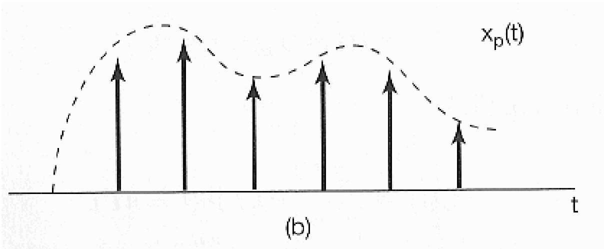
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\frac{w_c T}{\pi}\,\frac{\sin(w_c(t - nT))}{w_c(t - nT)}$$

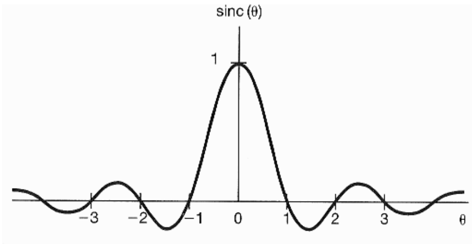
▪ Exact Interpolation:

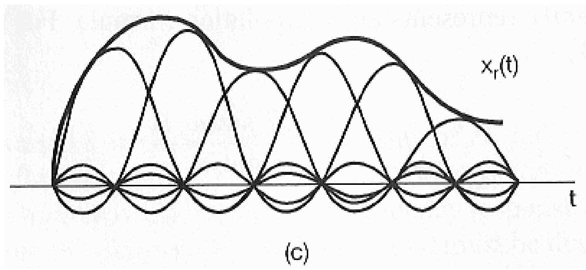




$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\frac{w_c T}{\pi}\,\frac{\sin(w_c(t-nT))}{w_c(t-nT)}$$

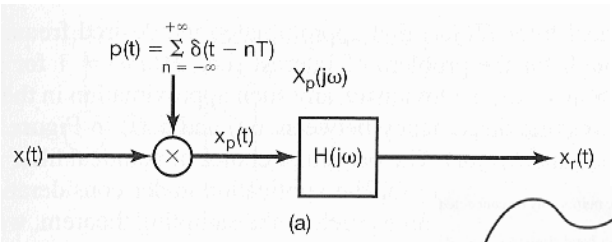


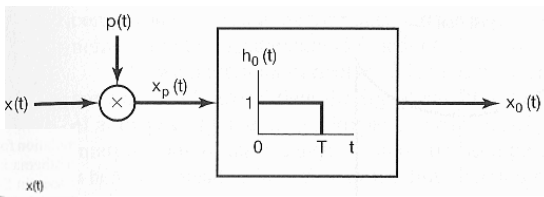


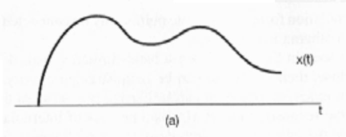


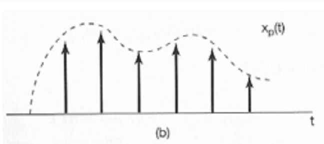
Reconstruction of a Signal from its Samples Using Interpolation
Shou shui Wei©2012

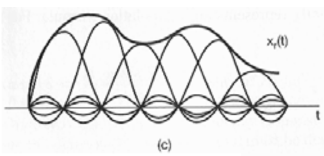
▪ Ideal Interpolating Filter & The Zero-Order Hold:

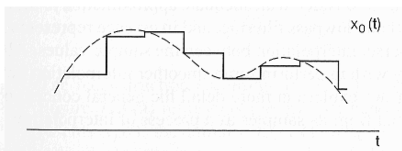


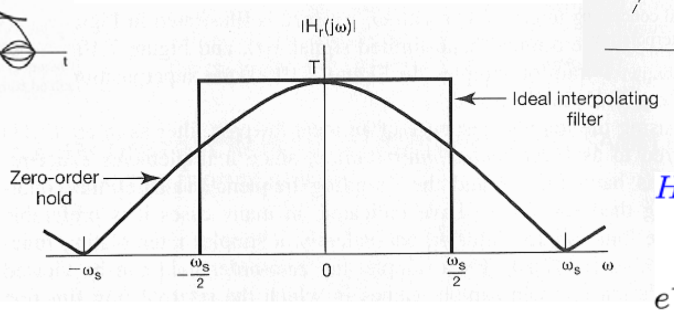










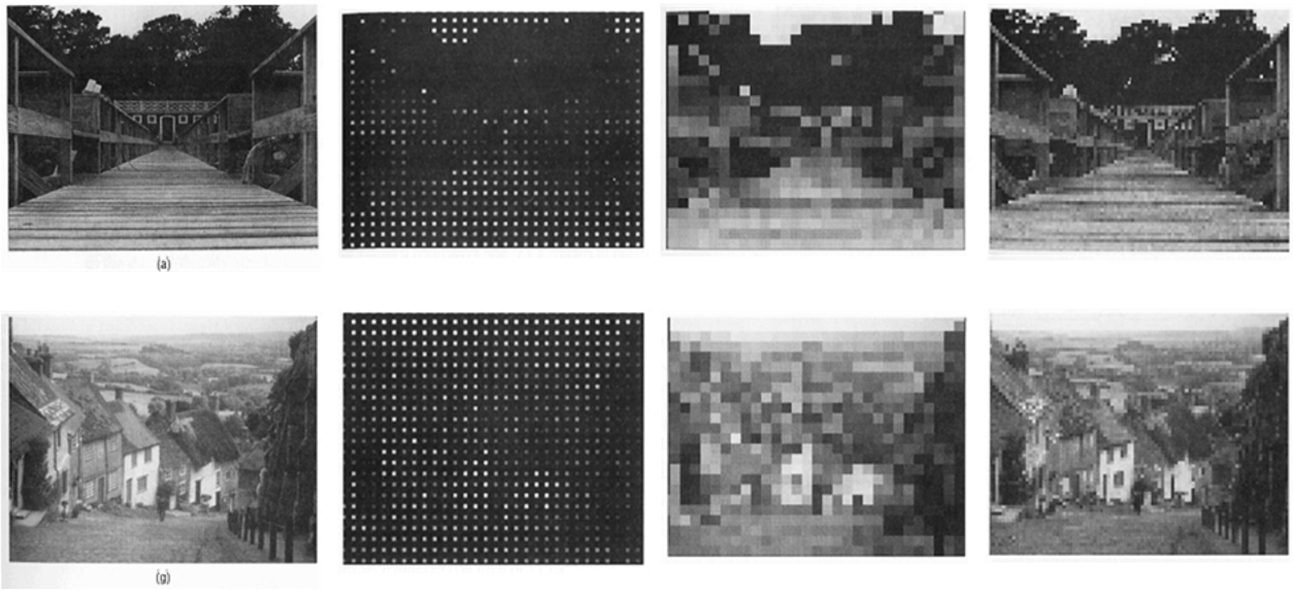


$$H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$

Sampling & Interpolation of Images:

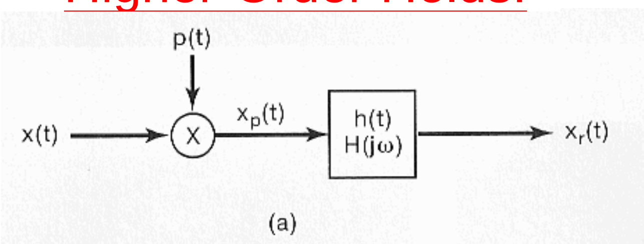
orignal image impulse sampling zero-order hold zero-order hold

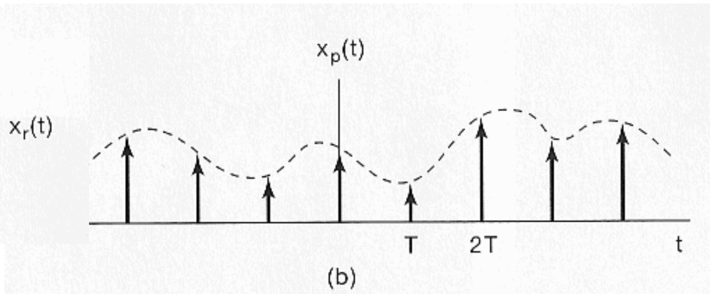
4 : 1

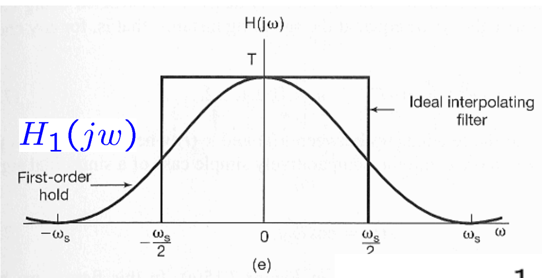


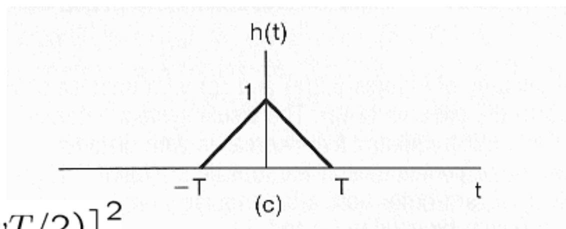
Reconstruction of a Signal from its Samples Using Interpolation
Shou shui Wei©2012

Higher-Order Holds:

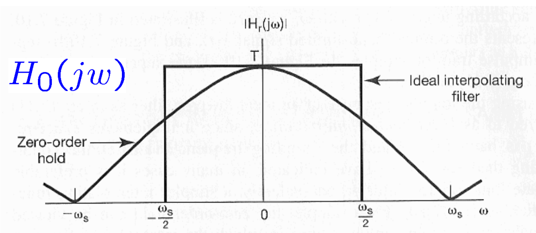


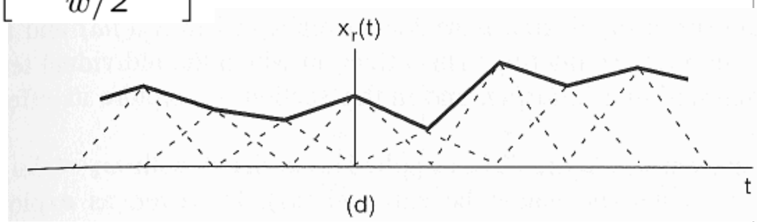






$$H_1(jw) = \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$

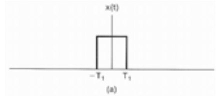




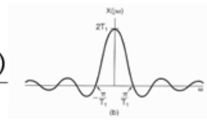
▪ Higher-Order Holds:

Ex 4.4, p. 293

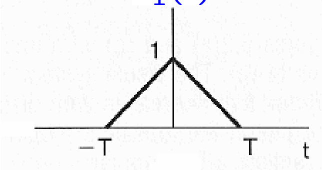
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



$$X(jw) = 2\frac{\sin(wT_1)}{w}$$

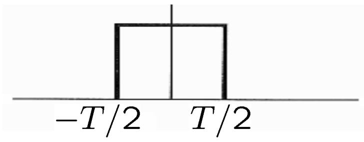


$h_1(t)$



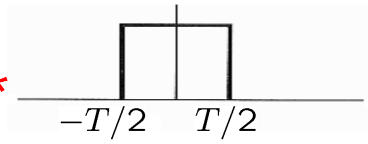
$$= \frac{1}{T}$$

$h_{sq}(t)$



*

$h_{sq}(t)$



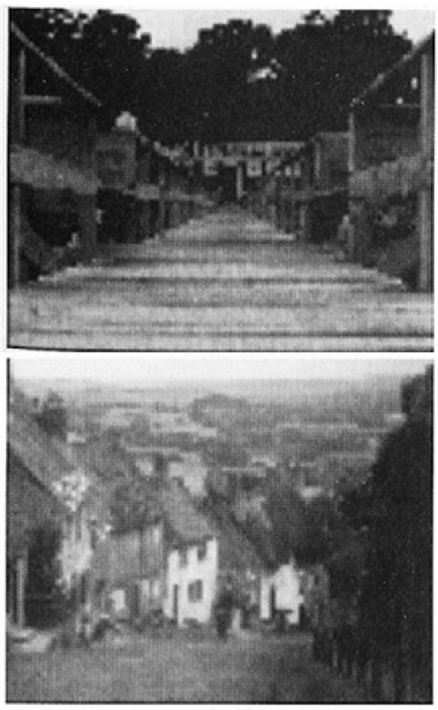
$$H_1(jw) = \frac{1}{T} \quad 2\frac{\sin(wT/2)}{w} \quad \text{X} \quad 2\frac{\sin(wT/2)}{w}$$

$$= \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$

▪ First-Order Hold on Image Processing:

zero-order hold

first-order hold

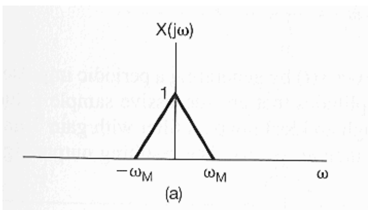


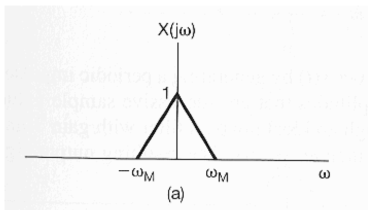
Shou shui Wei©2012

- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

Shou shui Wei©2012

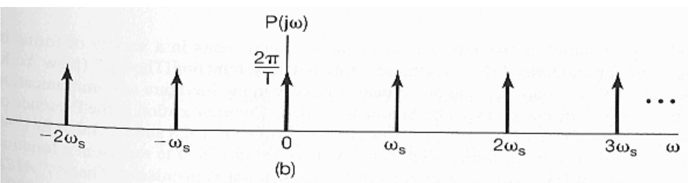
- Overlapping in Frequency-Domain: Aliasing

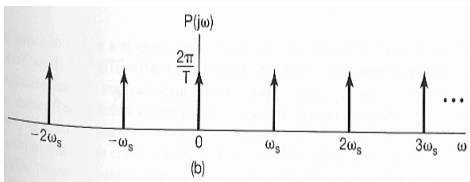


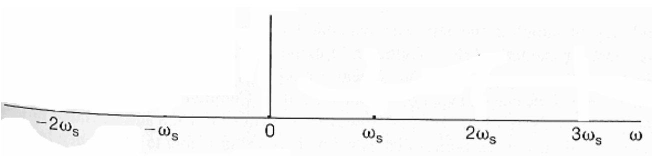


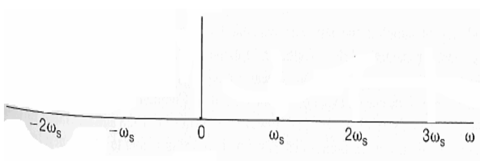
$w_s > 2w_M$

$w_s < 2w_M$

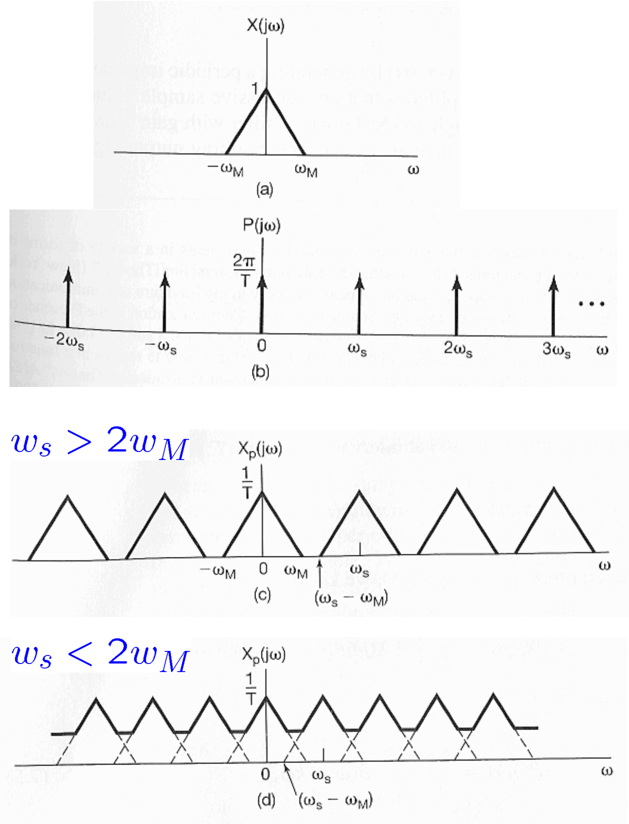






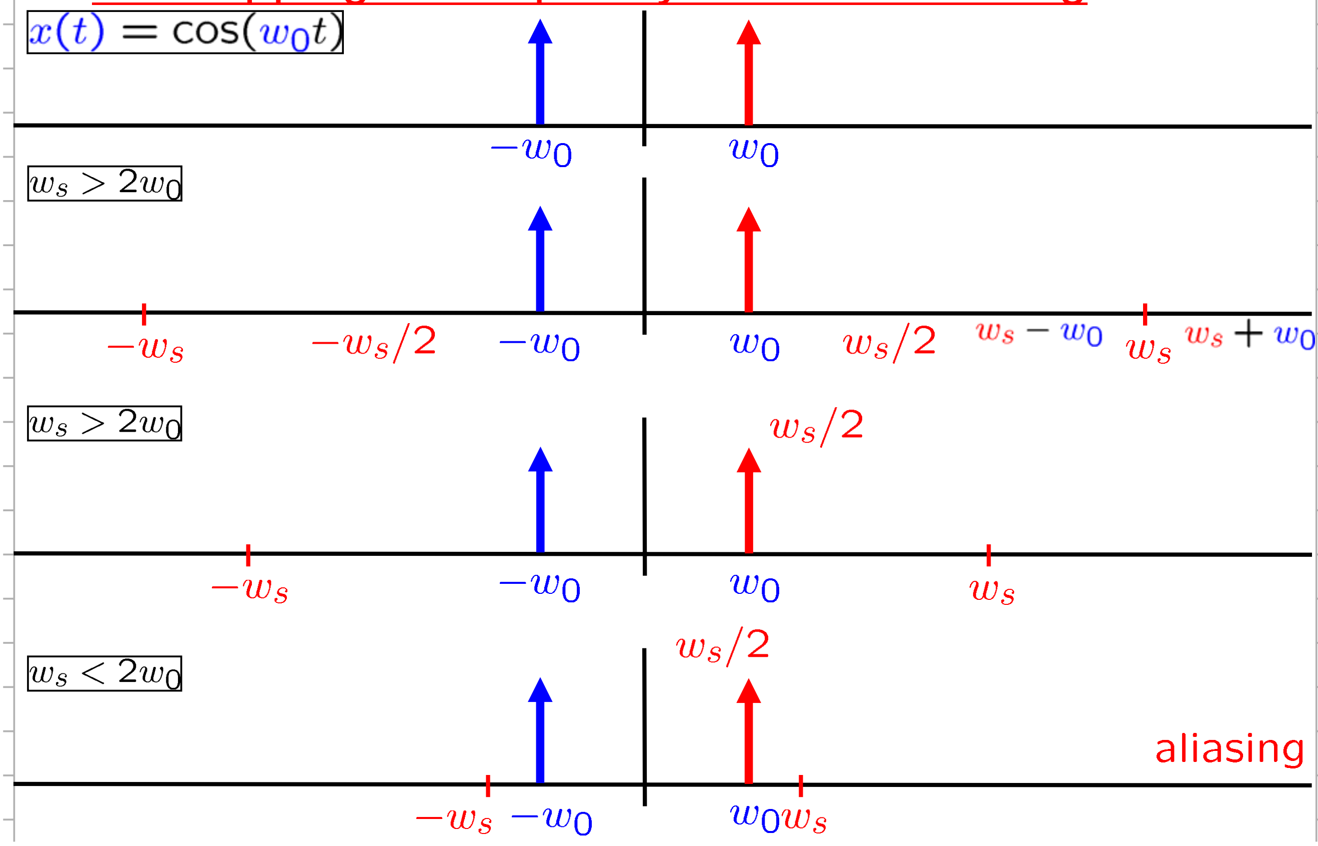


- Overlapping in Frequency-Domain: Aliasing



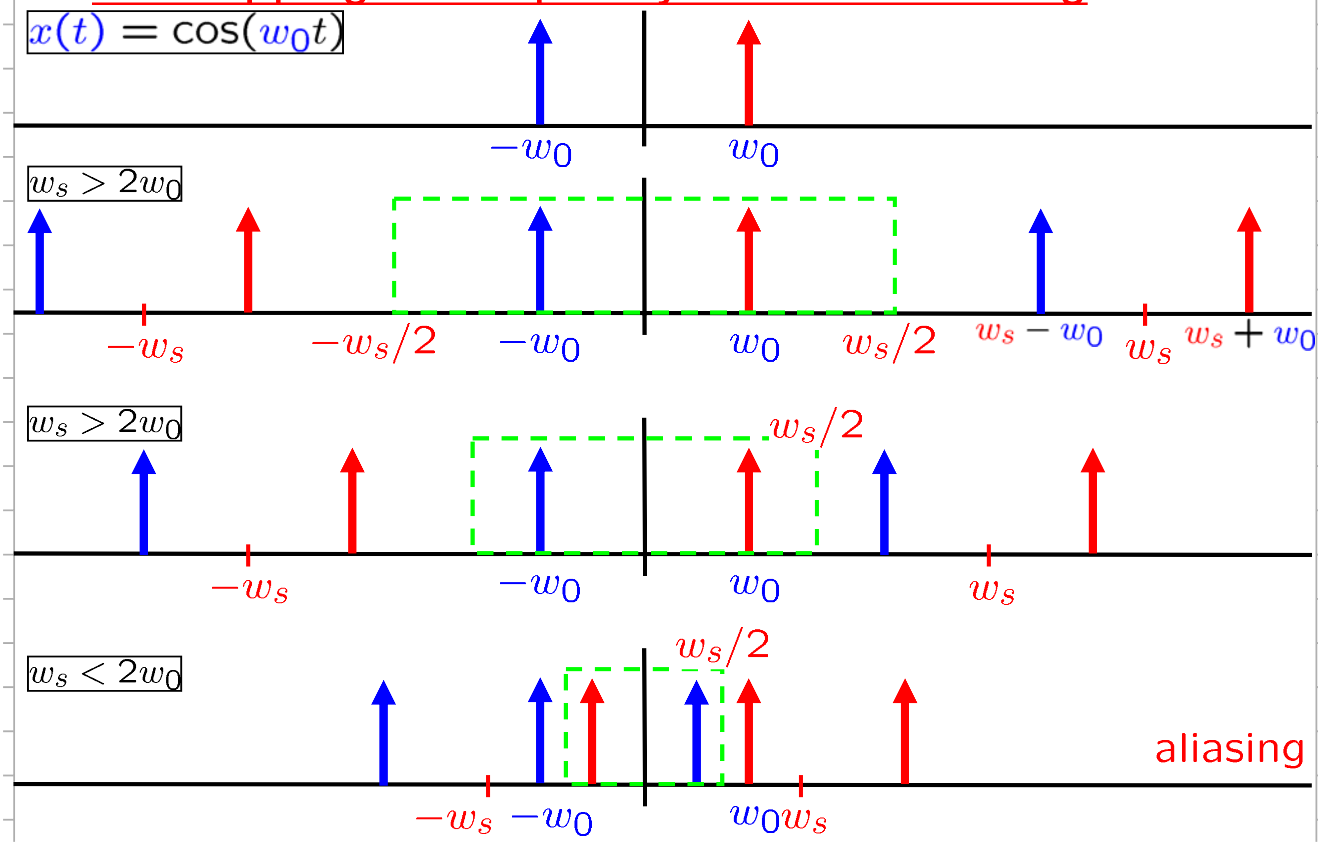
Effect of Under-sampling: Aliasing
Shou shui Wei©2012

- Overlapping in Frequency-Domain: Aliasing



Effect of Under-sampling: Aliasing

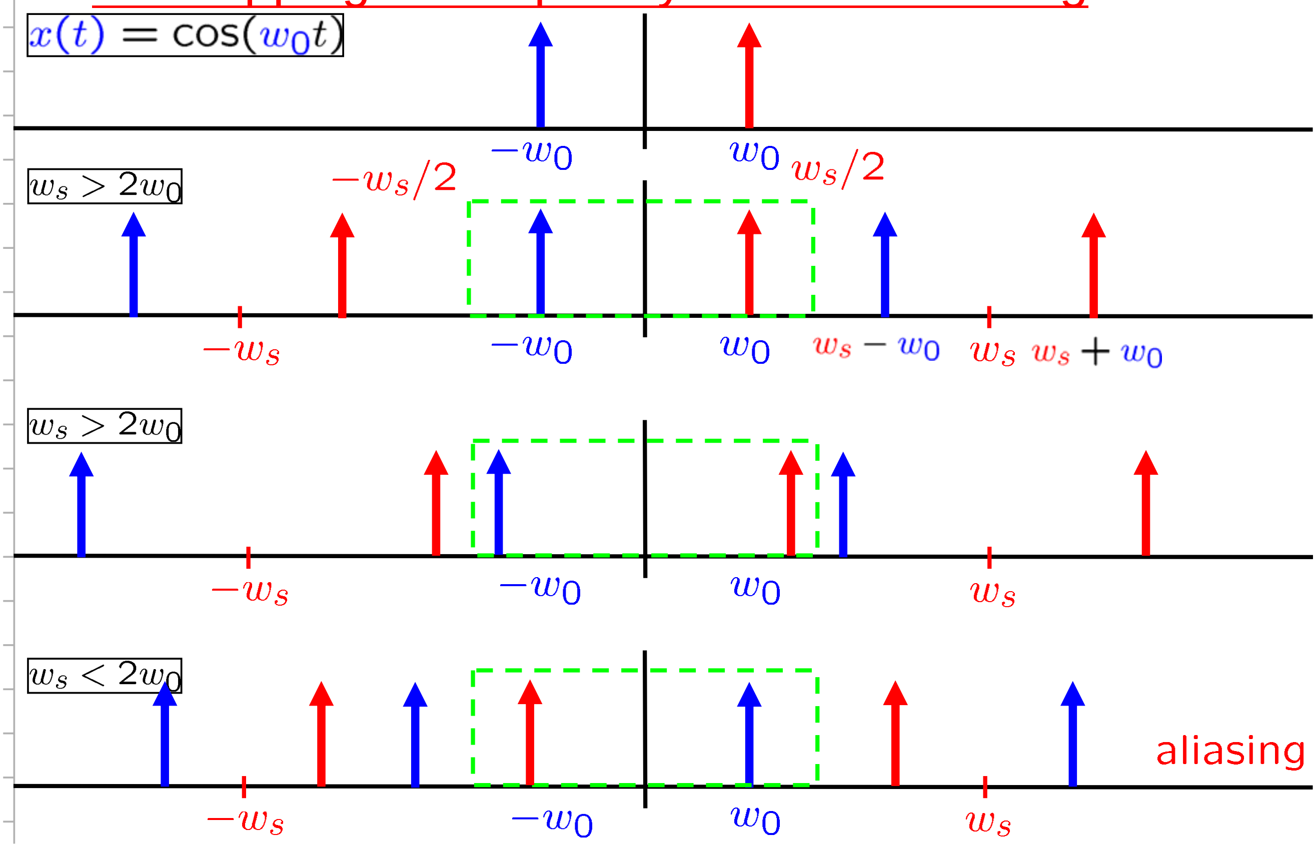
- Overlapping in Frequency-Domain: Aliasing



Effect of Under-sampling: Aliasing

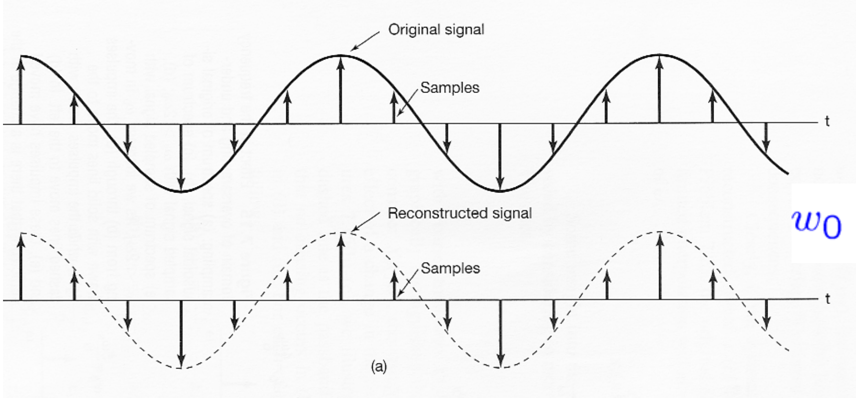
Shou shui Wei©2012

- Overlapping in Frequency-Domain: Aliasing

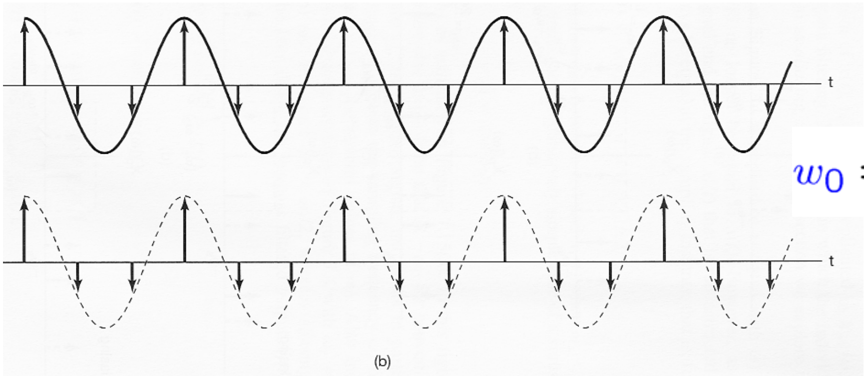


Shou shui Wei©2012

Overlapping in Frequency-Domain: Aliasing



$$w_0 = \frac{w_s}{6}$$

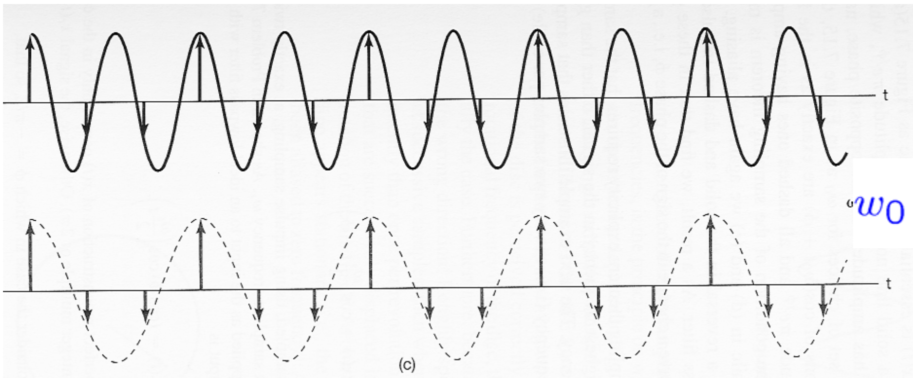


$$w_0 = \frac{2w_s}{6}$$

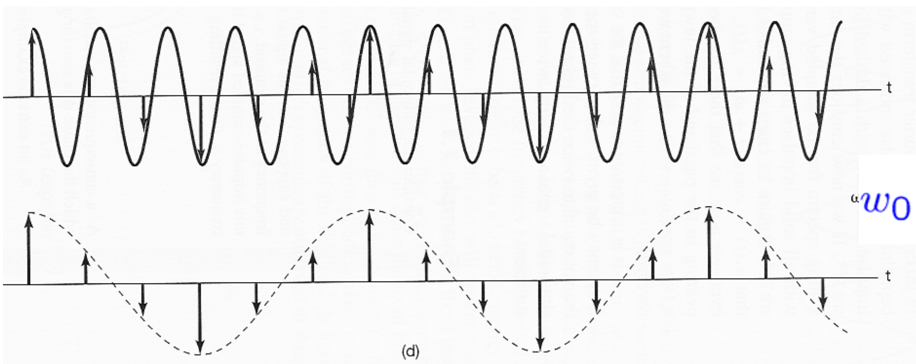
Effect of Under-sampling: Aliasing

Shou shui Wei©2012

Overlapping in Frequency-Domain: Aliasing



$$w_0 = \frac{4w_s}{6}$$

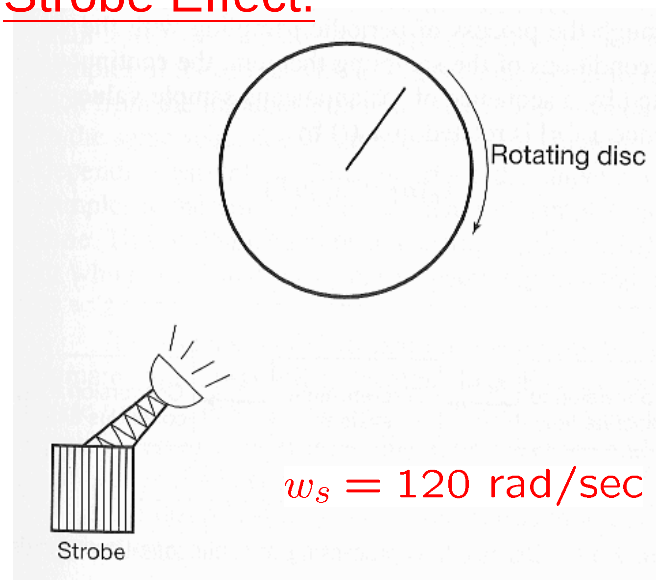


$$w_0 = \frac{5w_s}{6}$$

Effect of Under-sampling: Aliasing

Shou shui Wei©2012

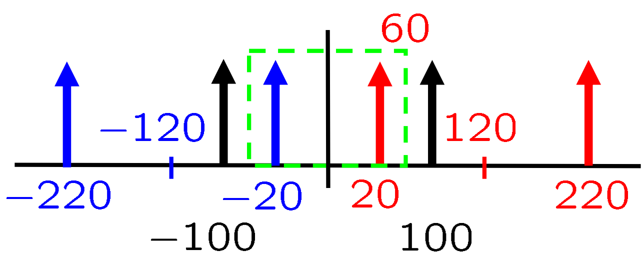
- Strobe Effect:



$w_0 = 100$ rad/sec

$w_s = 120$ rad/sec

$$\Rightarrow w = \pm w_s \pm w_0$$
$$= +20, -20$$

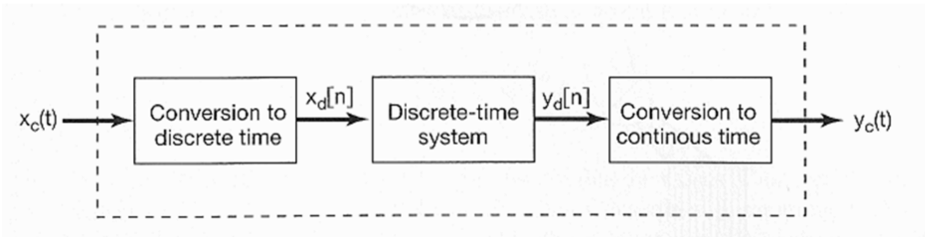


Outline

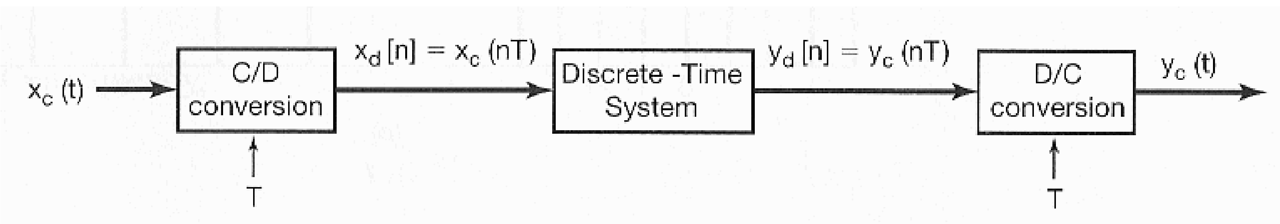
Shou shui Wei©2012

- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

- Discrete-Time Processing of CT Signals:



- C/D or A-to-D (ADC) and D/C or D-to-A (DAC):



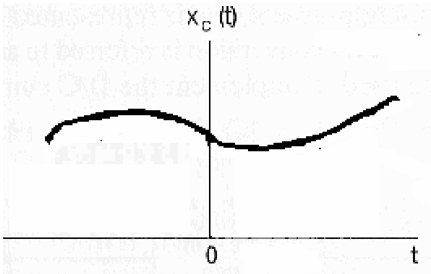
C/D: continous-to-discrete-time conversion

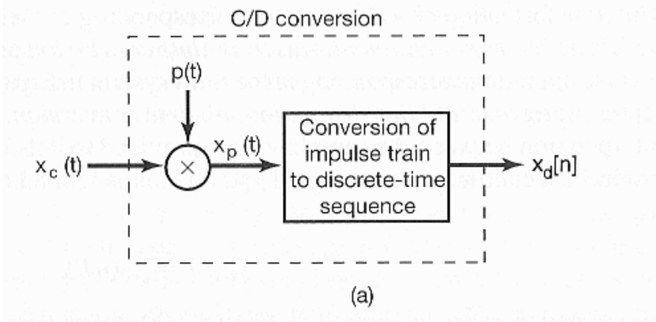
A-to-D: analog-to-digital converter

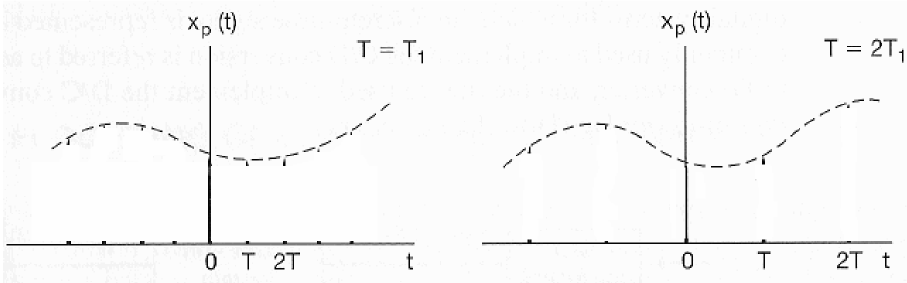
D/C: discrete-to-continous-time conversion

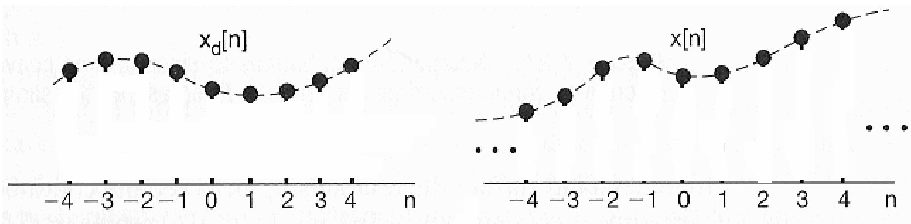
D-to-A: digital-to-analog converter

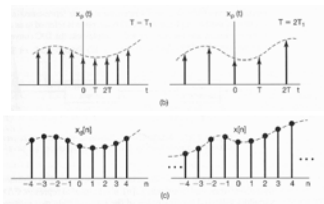
▪ C/D Conversion:



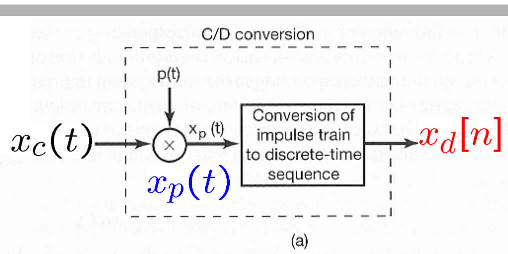


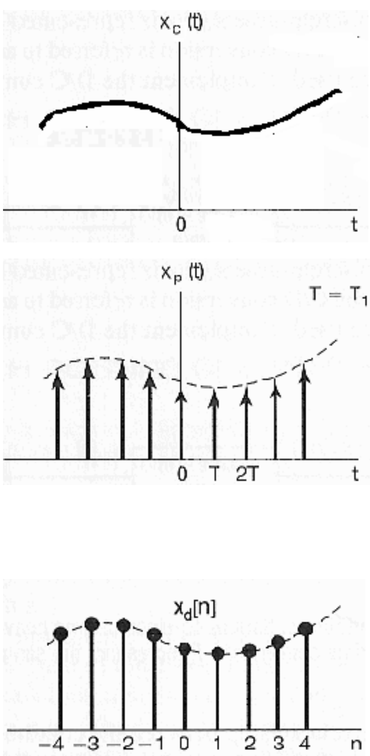


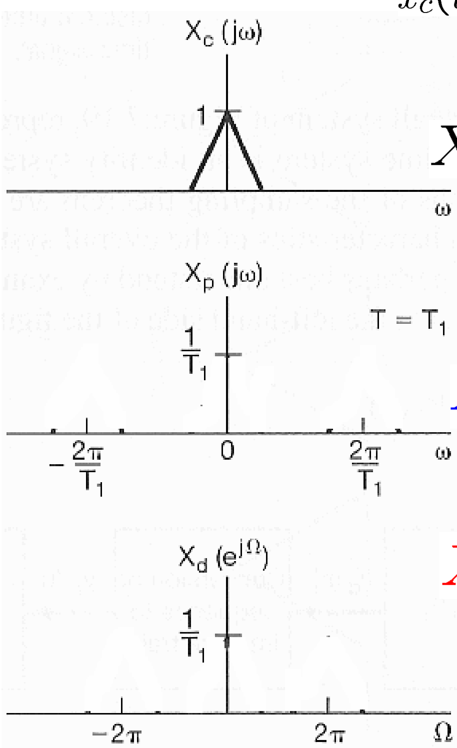




▪ C/D Conversion:



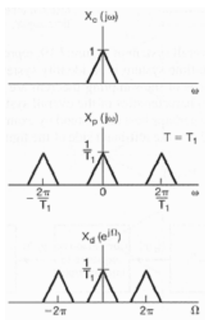




$X_c(jw)$

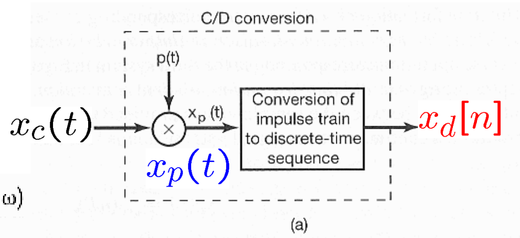
$X_p(jw)$

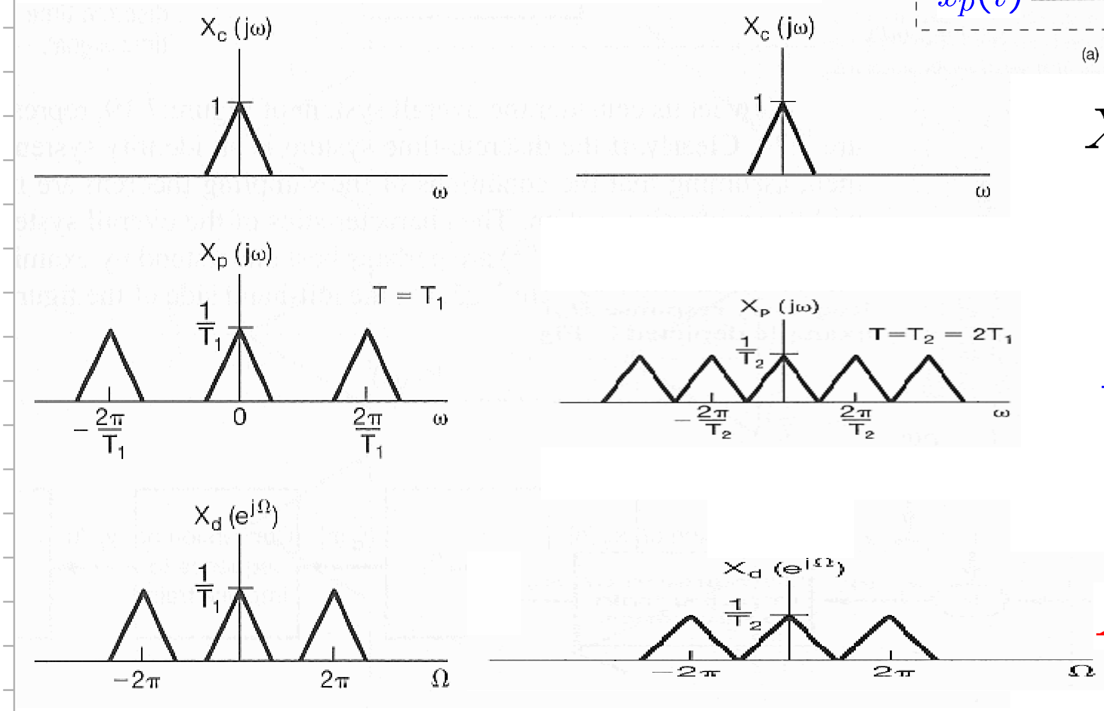
$X_d(e^{j\Omega})$



Shou shui Wei©2012

- C/D Conversion:





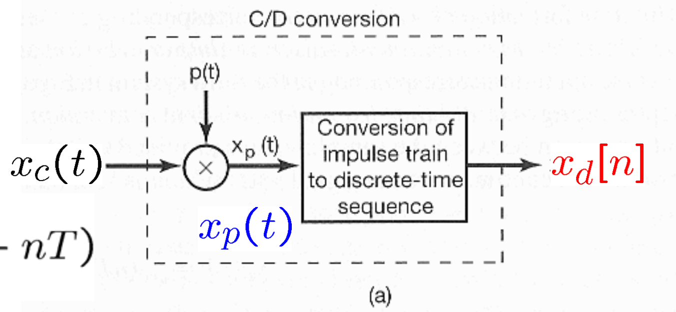
$X_c(jw)$

$X_p(jw)$

$X_d(e^{j\Omega})$

Shou shui Wei©2012

- C/D Conversion:



$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT)$$

Table 4.2, p. 329

$$\delta(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwt_0}$$

Eq 7.3, 7.6, p. 517

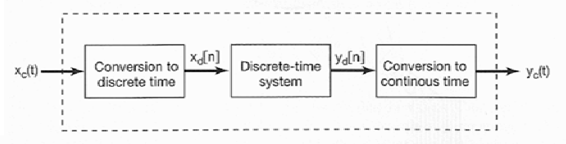
$$X_p(jw) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-jwnT} = \frac{1}{T}\sum_{K=-\infty}^{+\infty} X_c(j(w - kw_s))$$

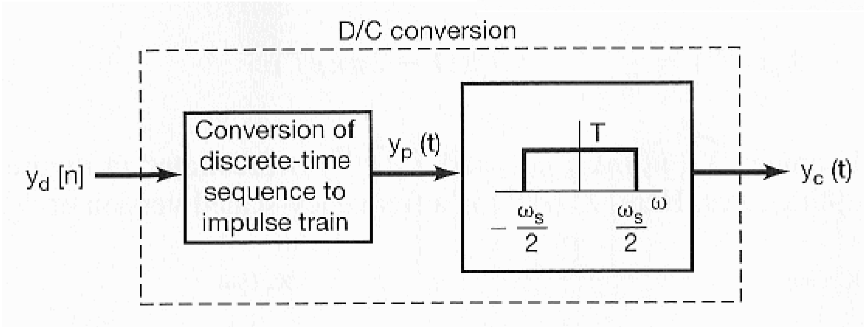
$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n}$$

$$\Rightarrow X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right) = \frac{1}{T}\sum_{K=-\infty}^{+\infty} X_c\left(j\left(\frac{\Omega}{T} - k\frac{2\pi}{T}\right)\right)$$

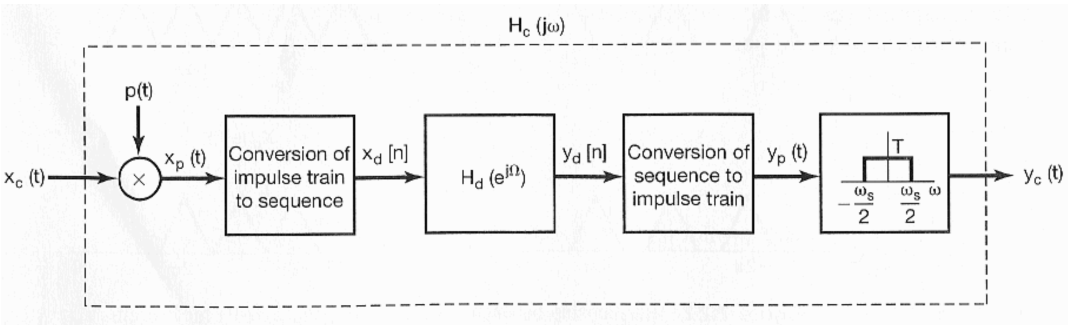
Shou shui Wei©2012

▪ D/C Conversion:



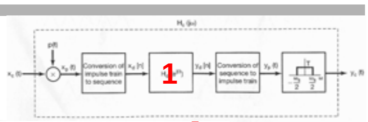


▪ Overall System:

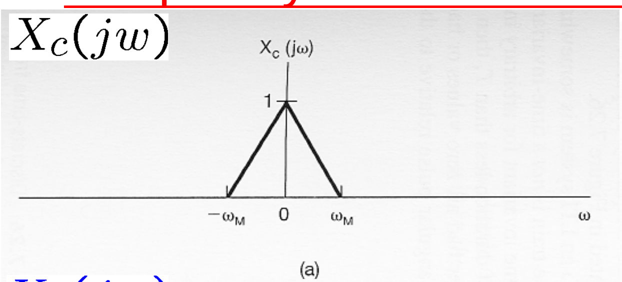


Shou shui Wei©2012

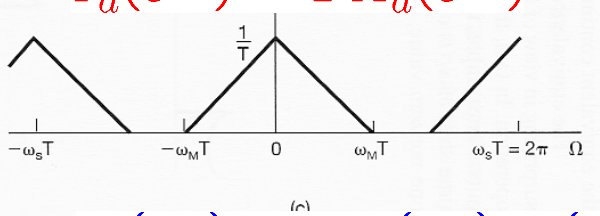
▪ Frequency-Domain Illustration:



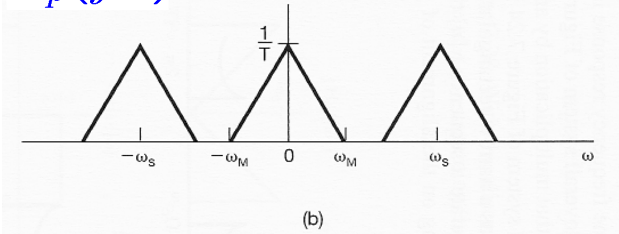
$X_c(jw)$



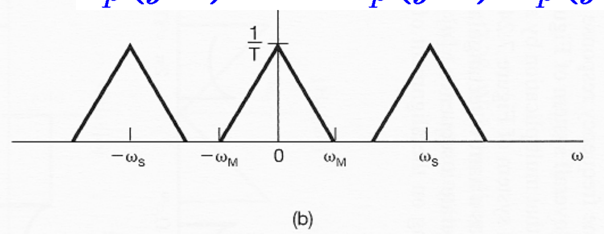
$Y_d(e^{j\Omega}) = 1\ X_d(e^{j\Omega})$



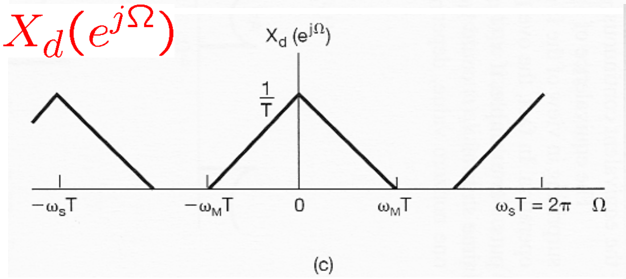
$X_p(jw)$



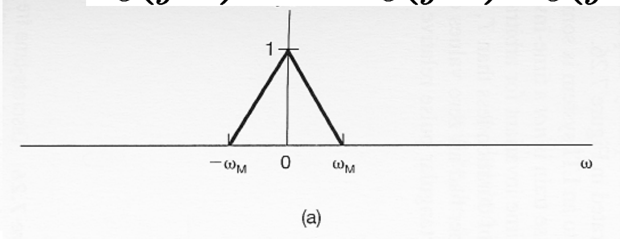
$Y_p(jw) \ = \ H_p(jw)X_p(jw)$



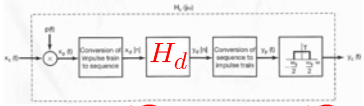
$X_d(e^{j\Omega})$



$Y_c(jw) \ = \ H_c(jw)X_c(jw)$



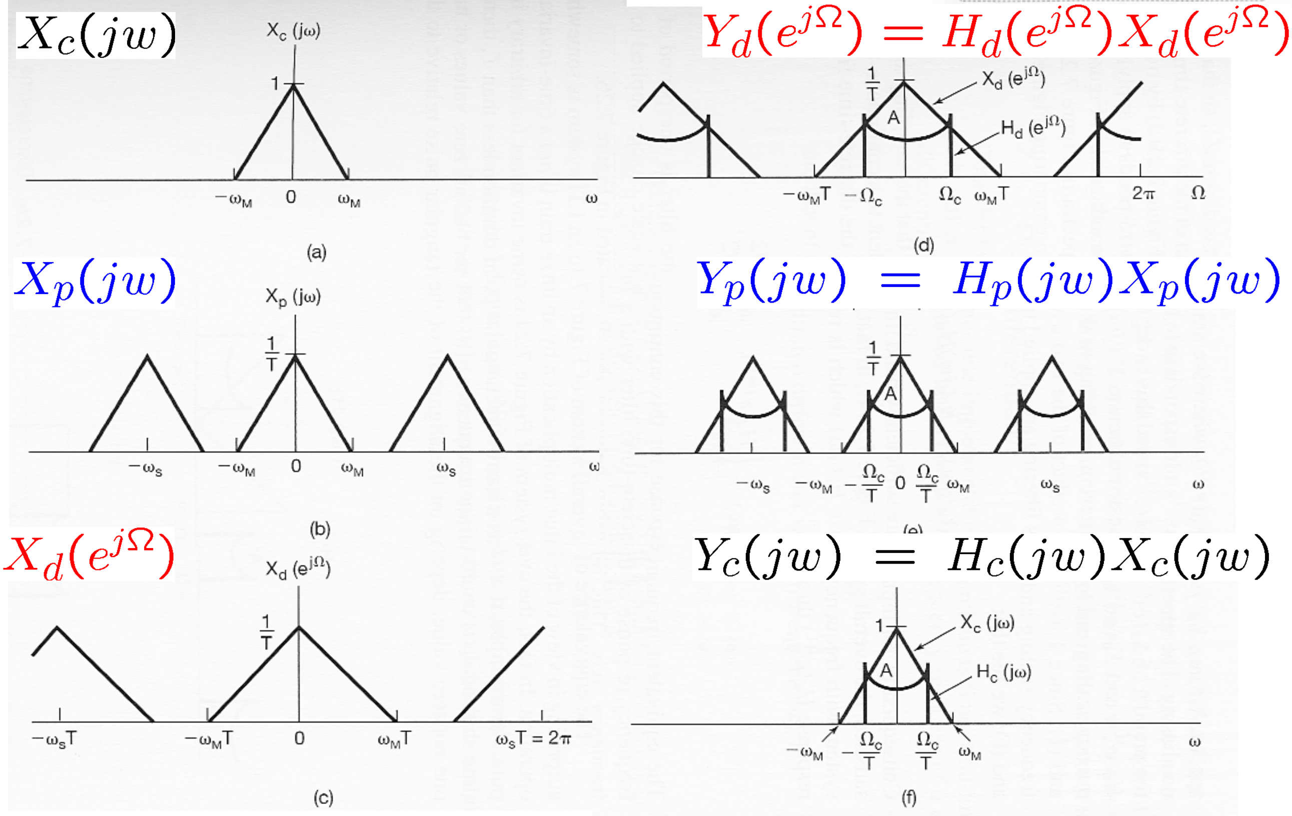
▪ Frequency-Domain Illustration:



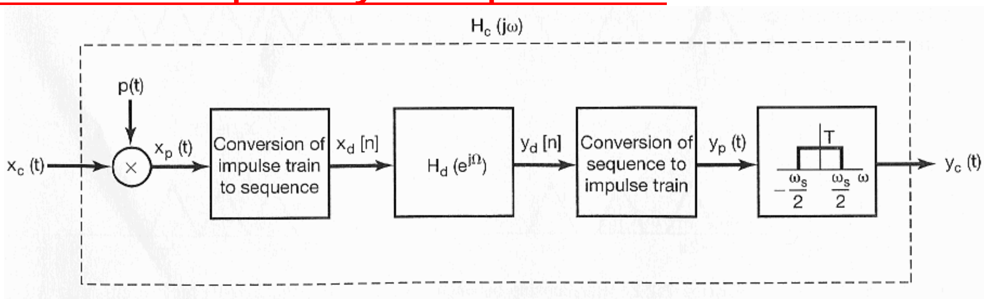
$X_c(jw)$ $\quad$ $Y_d(e^{j\Omega}) = H_d(e^{j\Omega})X_d(e^{j\Omega})$

$X_p(jw)$ $\quad$ $Y_p(jw) = H_p(jw)X_p(jw)$

$X_d(e^{j\Omega})$ $\quad$ $Y_c(jw) = H_c(jw)X_c(jw)$

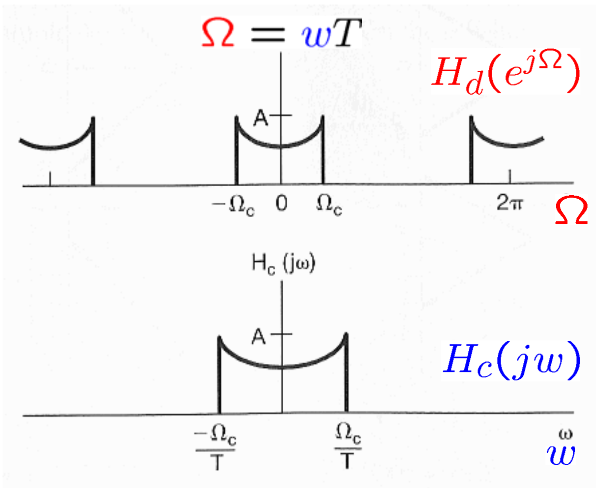


▪ CT & DT Frequency Responses:



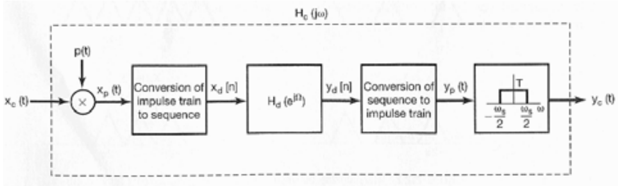
$$Y_c(jw) = X_c(jw)H_c(jw)$$

$$H_c(jw) = \begin{cases} H_d(e^{jwT}), & |w| < w_s/2 \\ 0, & |w| > w_s/2 \end{cases}$$



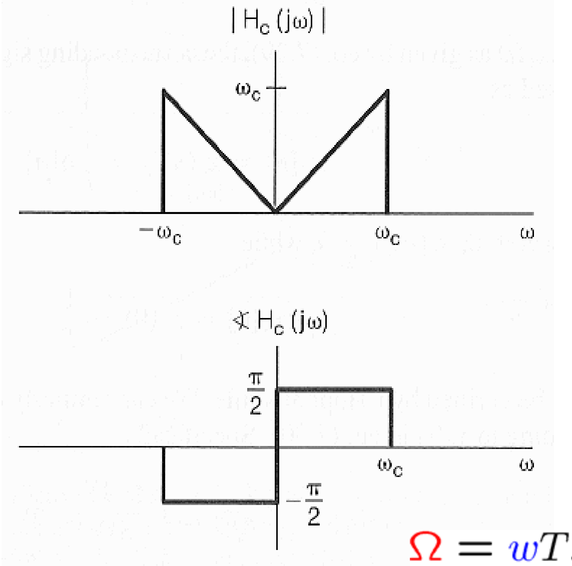
Digital Differentiator:

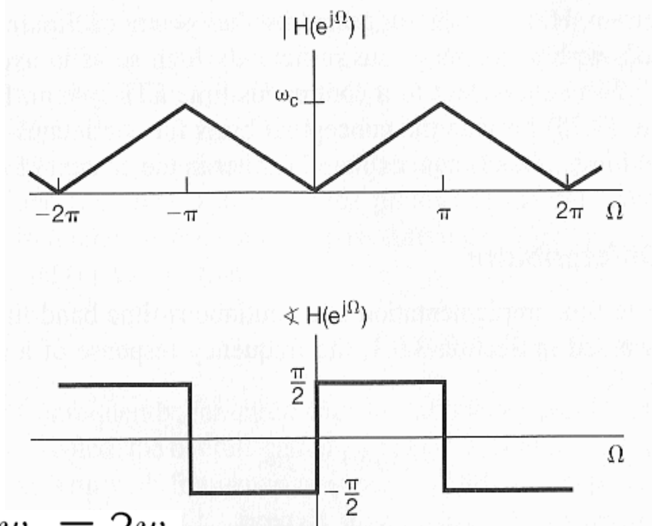
Ex 4.16, p. 317



$$H_c(jw) = \begin{cases} jw, & |w| < w_c \\ 0, & |w| > w_c \end{cases}$$

$$H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right), \quad |\Omega| < \pi$$

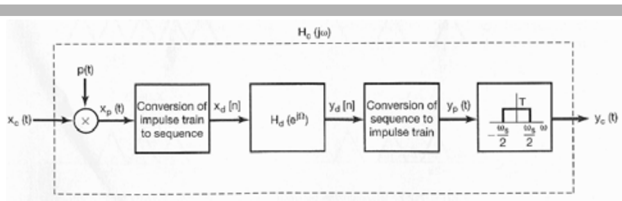




$$\Omega = wT, \ w_s = 2w_c$$

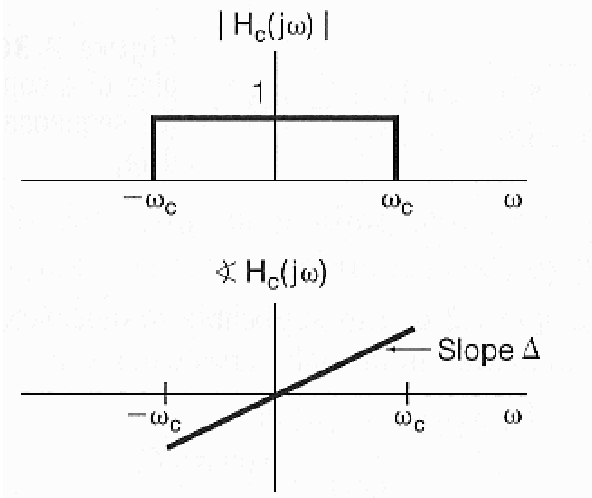
Half-Sample Delay:

Ex 4.15, p. 317

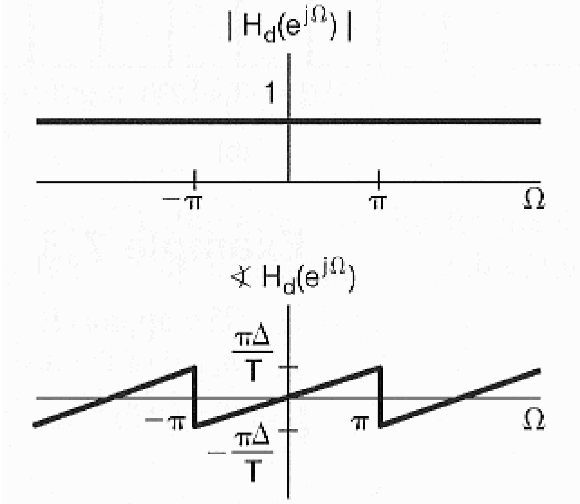


$$H_c(jw) = \begin{cases} e^{-jw\Delta}, & |w| < w_c \\ 0, & |w| > w_c \end{cases}$$

$$H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}, \quad |\Omega| < \pi$$



(a)



(b)

$$\Omega = wT, \ w_s = 2w_c$$

Outline

- Representation of of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Undersampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals
- Sampling of Discrete-Time Signals

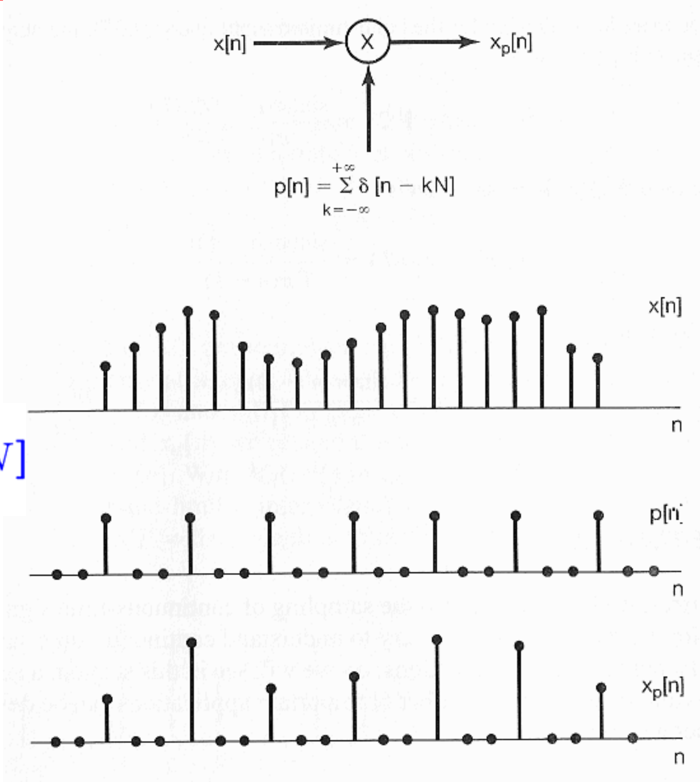
Sampling of Discrete-Time Signals

- Impulse-Train Sampling:

$$x_p[n] = \begin{cases} x[n], & \text{if } n = kN \\ 0, & \text{otherwise} \end{cases}$$

$$x_p[n] = x[n]\,p[n]$$

$$= \sum_{k=-\infty}^{+\infty} x[kN]\,\delta[n-kN]$$

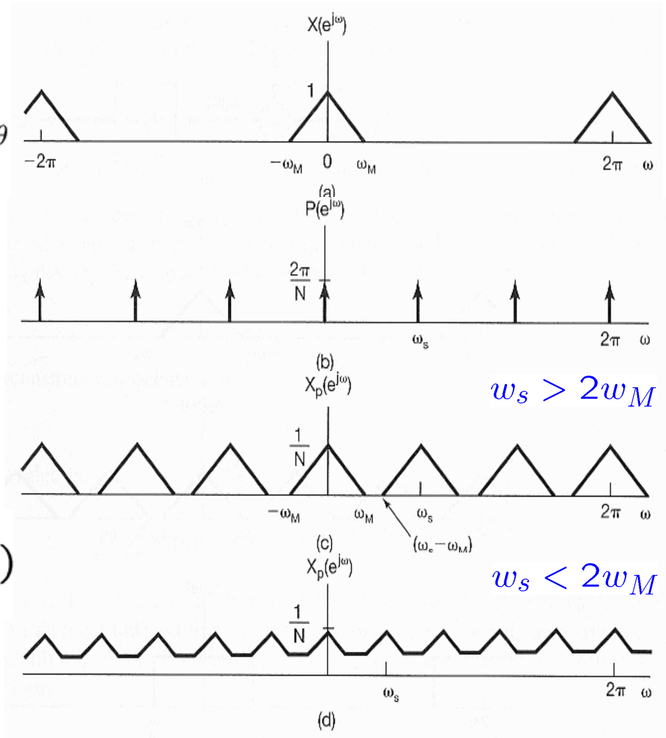


Impulse-Train Sampling:

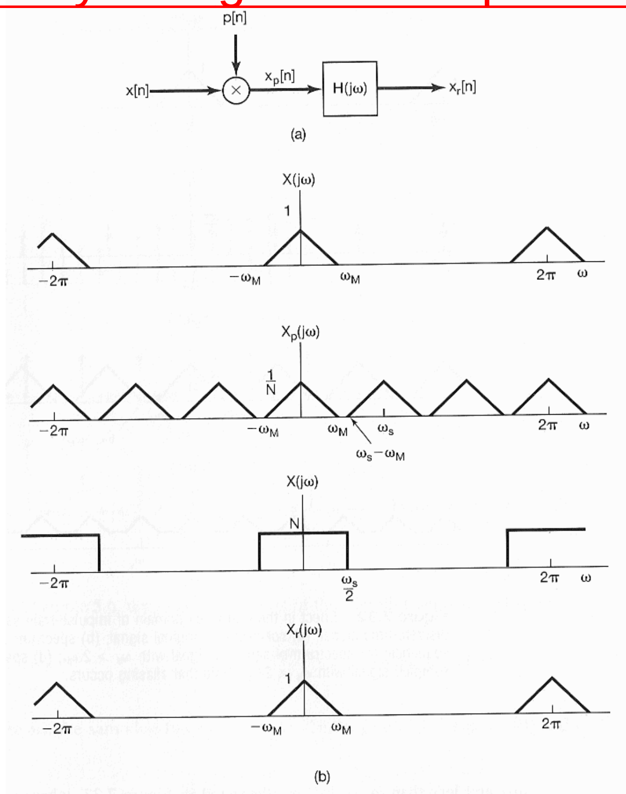
$$X_p(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(w-\theta)})d\theta$$

$$P(e^{jw}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(w-kw_s)$$

$$\Rightarrow X_p(e^{jw}) = \frac{1}{N}\sum_{k=0}^{N-1}X(e^{j(w-kw_s)})$$

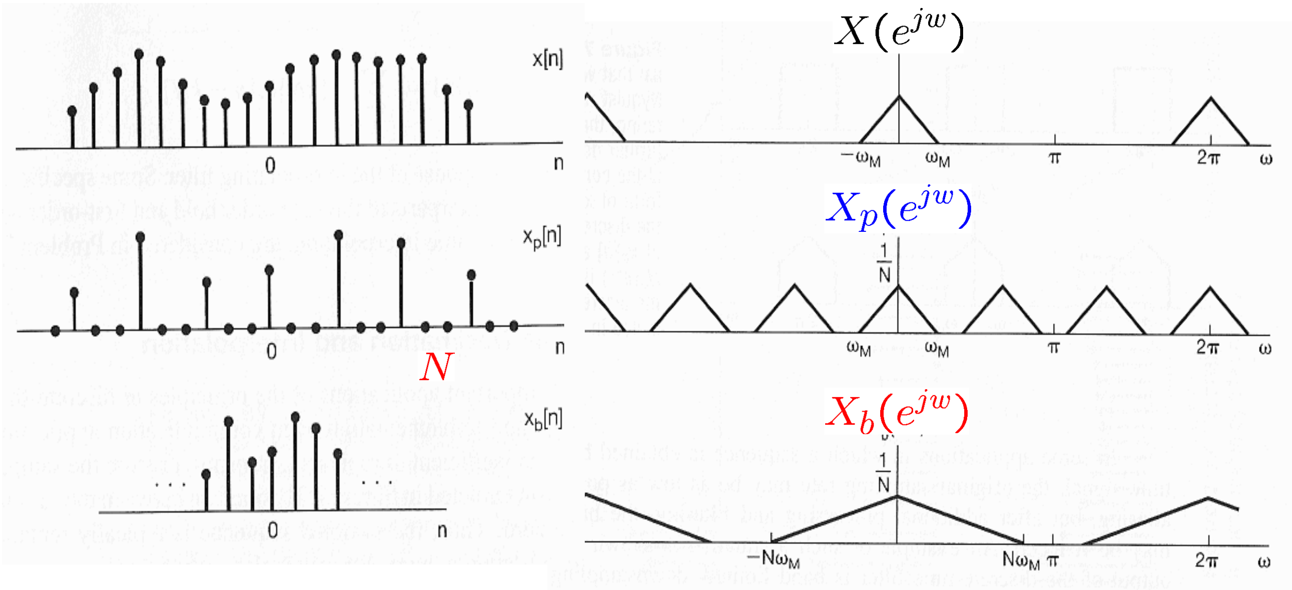


Exact Recovery Using Ideal Lowpass Filter:



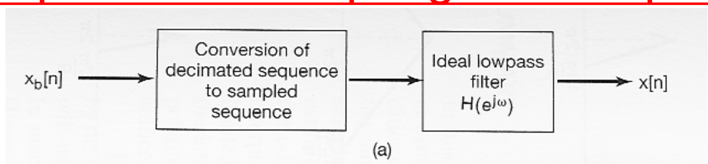
▪ DT Decimation & Interpolation: Down-sampling

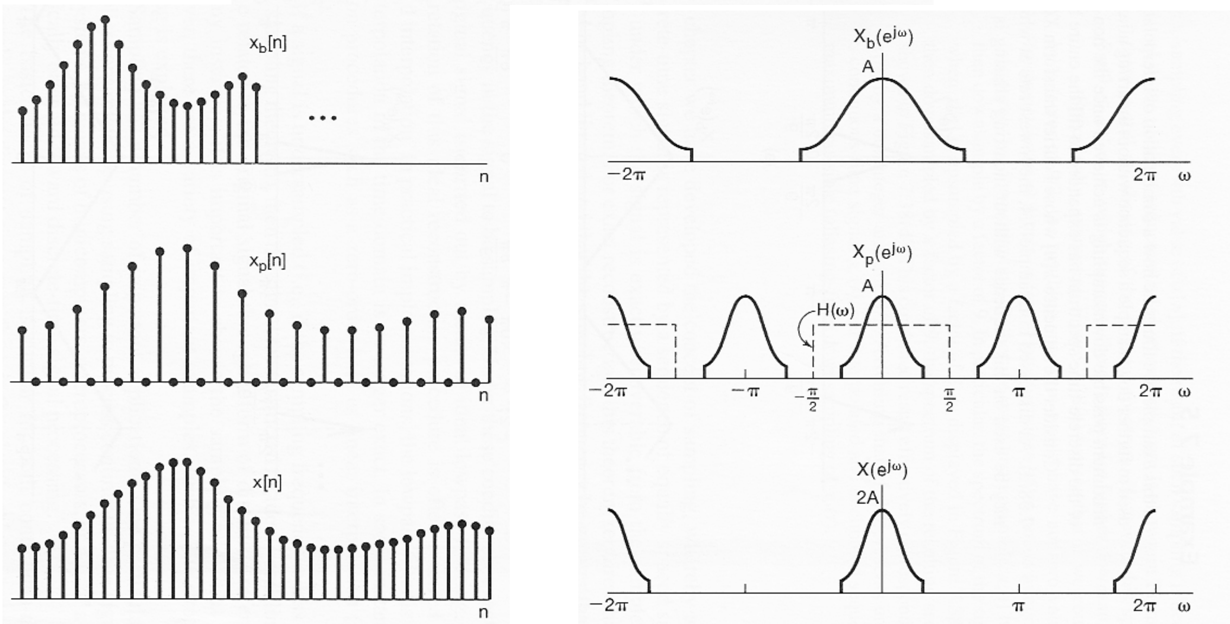
Eq 5.45, p. 378: Time expansion



$$X_b(e^{jw}) = X_p(e^{jw/N})$$

▪ Higher Equivalent Sampling Rate: Up-sampling





Shou shui Wei©2012

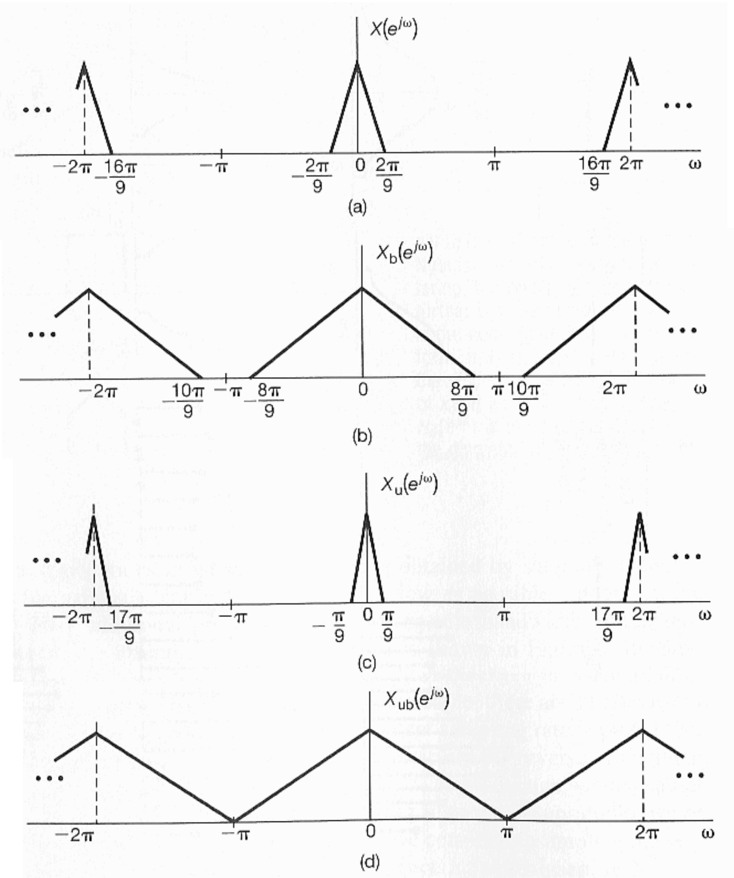
- Down-sampling + Up-sampling:

$$\frac{2\pi}{9} \times 4 < \pi$$

$$\frac{2\pi}{9} \times \frac{9}{2} = \pi$$

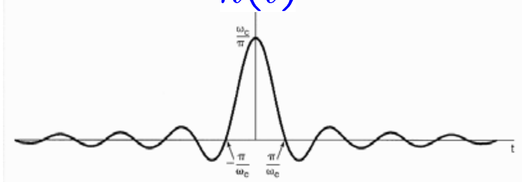
$$\frac{2\pi}{9} \times \frac{1}{2} = \frac{\pi}{9}$$

$$\frac{\pi}{9} \times 9 = \pi$$

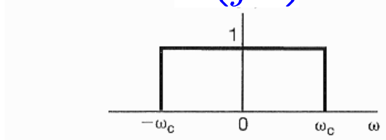


Shou shui Wei©2012

$h(t)$



$H(jw)$



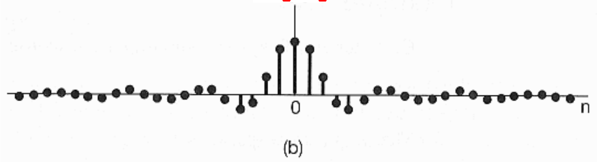
w

$$h(t) \xleftrightarrow{\mathcal{C.T.F.T.}} H(jw)$$

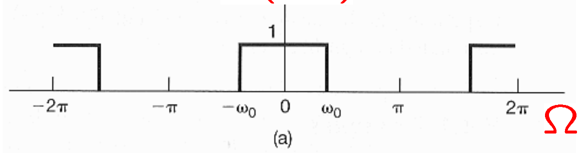
$$w_s = \frac{2\pi}{T} \qquad \Omega = wT$$

$$h[n] \xleftrightarrow{\mathcal{D.T.F.T.}} H(e^{j\Omega})$$

$h[n]$



$H(e^{j\Omega})$



Ω

Shou shui Wei©2012

- Discrete-Time Processing of CT Signals